Theorems on the reduction of integral formulas to the quadrature of the cirlce*

Leonhard Euler

Lemma 1

§1 *In the circle whose radius is* $= 1$ *and the half of whose circumference is* $= \pi$ *let the sine of a certain angle* s *be* $= x$*; the same quantity* x *will be the sine of all arcs contained in this infinite series*

$$s, \quad \pi - s, \quad 2\pi + s, \quad 3\pi - s, \quad 4\pi + s, \quad 5\pi - s \quad \text{etc.}$$

Furthermore, x *will be the sine of all negative arcs contained in this series*

$$-\pi - s, \quad -2\pi + s, \quad -3\pi - s, \quad -4\pi + s, \quad -5\pi - s \quad \text{etc.}$$

Corollary 1

§2 Therefore, if i denotes an arbitrary positive number, so all arcs contained in this expression $\pm 2i\pi + s$ as the arcs contained in this expression $\pm(2i+1)\pi - s$ will have the same sine x.

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COROLLARY 2

§3 Since the sines of negative angles are also negative, the sine of all angles contained in this form $\pm 2i\pi - s$ are $= -x$ and also the sines of all angles contained in this form $\pm(2i+1)\pi + s$ are $= -x$, if the sine of the angle s was $= +x$, of course.

LEMMA 2

§4 *In the circle whose radius is $= 1$ and the half of whose circumference is $= \pi$ let the cosine of a certain angle s be $= y$; the same quantity y will be the cosine of all arcs contained in this infinite series*

$$s, \quad 2\pi - s, \quad 2\pi + s, \quad 4\pi - s, \quad 4\pi + s, \quad 6\pi - s \quad \text{etc.}$$

and likewise the same quantity y will be the cosine of all negative angles contained in this series

$$-s, \quad -2\pi + s, \quad -2\pi - s, \quad -4\pi + s, \quad -4\pi - s, \quad -6\pi + s \quad \text{etc.}$$

COROLLARY 1

§5 Therefore, if i denotes an arbitrary integer number, all arcs contained in this general expression $\pm 2i\pi \pm s$ will be same and $= y$.

COROLLARY 2

§6 Since the cosine of the angle, if it augmented or diminished by two right ones or π, becomes negative, all angles contained in this form $\pm(2i+1)\pi \pm s$ will have the same cosine $= y$, if the cosine of the angle s was $= +y$.

LEMMA 3

§7 *If the tangent of the angle s is t, t will also be the tangent of all so positive as negative angles contained in these two series*

$$s, \quad \pi + s, \quad 2\pi + s, \quad 3\pi + s, \quad 4\pi + s, \quad 5\pi + s \quad \text{etc.},$$
$$-\pi + s, \quad -2\pi + s, \quad -3\pi + s, \quad -4\pi + s, \quad -5\pi + s \quad \text{etc.}$$

COROLLARY

§8 Therefore, while i denotes an arbitrary number, all angles contained in this expression $\pm i\pi + s$ will have the same tangent $= t$; but the tangent of the angles $\pm i\pi - s$ will be $= -t$, if the tangent of the angle s is $= +t$, of course.

PROBLEM 1

§9 *To find the roots of this infinite equation*

$$x = z - \frac{z^3}{1\cdot 2\cdot 3} + \frac{z^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{z^7}{1\cdot 2\cdot 3\cdots 7} + \text{etc.}$$

SOLUTION

If z denotes the arc of the circle whose radius is $= 1$, x will be the sine of that arc. Therefore, let us put the arc, whose sine $= x$, to be s; the infinitely many values of z will be contained in these two series

$$s, \quad \pi - s, \quad 2\pi + s, \quad 3\pi - s, \quad 4\pi + s, \quad 5\pi - s \quad \text{etc.},$$

$$-\pi - s, \quad -2\pi + s, \quad -3\pi - s, \quad -4\pi + s, \quad -5\pi - s \quad \text{etc.}$$

Q. E. I.

COROLLARY 1

§10 Therefore, if the propounded equation is transformed into this form

$$0 = 1 - \frac{z}{1\,x} + \frac{z^3}{1\cdot 2\cdot 3\,x} - \frac{z^5}{1\cdot 2\cdot 3\cdot 4\cdot 5\,x} + \frac{z^7}{1\cdot 2\cdots 7\,x} - \text{etc.},$$

one will have the following infinitely many factors of it

$$\left(1-\frac{z}{s}\right)\left(1-\frac{z}{\pi-s}\right)\left(1-\frac{z}{\pi+s}\right)\left(1-\frac{z}{2\pi-s}\right)\left(1-\frac{z}{2\pi+s}\right)\left(1-\frac{z}{3\pi-s}\right)\left(1-\frac{z}{3\pi+s}\right)\text{etc.};$$

the structure of the progression in the factors is immediately seen.

COROLLARY 2

§11 Therefore, since the coefficients of the second term in the product become equal to the sum of the coefficients of z in all factors, it will be

$$\frac{1}{x} = \frac{1}{s} + \frac{1}{\pi - s} - \frac{1}{\pi + s} - \frac{1}{2\pi - s} + \frac{1}{2\pi + s} + \frac{1}{3\pi - s} - \text{etc.}$$

COROLLARY 3

§12 Let $s = \frac{m}{n}\pi$ such that it is $x = \sin\frac{m}{n}\pi$; having multiplied the series by $\frac{\pi}{n}$ it will be

$$\frac{\pi}{nx} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \text{etc.}$$

COROLLARY 4

§13 Since the coefficient of z^2, which is $=0$, in the product becomes equal to the sum the products consisting of two coefficients of z in the factors, but on the other hand this sum taken twice is equal to the square of the sum of these coefficients less the sum of the same squares, it will be

$$\frac{1}{xx} = \frac{1}{ss} + \frac{1}{(\pi - s)^2} + \frac{1}{(\pi + s)^2} + \frac{1}{(2\pi - s)^2} + \frac{1}{(2\pi + s)^2} + \text{etc.}$$

COROLLARY 5

§14 Therefore, having again put $s = \frac{m}{n}\pi$ that it is $x = \sin\frac{m}{n}\pi$, this summation will result

$$\frac{\pi\pi}{nnxx} = \frac{1}{m^2} + \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \text{etc.}$$

SCHOLIUM

§15 I could proceed further this way and determine the sum of the higher powers; but since I already did this in another paper[1] and these series suffice

[1]Euler refers to his paper "De summis serierum reciprocarum ex potestatibus numerorum naturalium ortarum dissertatio altera, in qua eaedem summationes ex fonte maxime diverso derivantur". This is E61 in the Eneström-Index.

for our undertaking, I will omit the further investigation of these series here.

PROBLEM 2

§16 *To find the roots of this infinite equation*

$$y = 1 - \frac{z^2}{1 \cdot 2} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{z^6}{1 \cdot 2 \cdots 6} + \frac{z^8}{1 \cdot 2 \cdots 8} - \text{etc.}$$

SOLUTION

If z denotes the arc on the circle whose radius is $= 1$, y will be the cosine of this arc. Therefore, if one takes the arc s, whose cosine is $= y$, innumerable values of z will be contained in the following two series

$$\begin{array}{c} s, \quad 2\pi - s, \quad 2\pi + s, \quad 4\pi - s, \quad 4\pi + s \quad \text{etc.} \\ -s, \quad -2\pi + s, \quad -2\pi - s, \quad -4\pi + s, \quad -4\pi - s \quad \text{etc.} \end{array}$$

Q. E. I.

COROLLARY 1

§17 Therefore, if the propounded equation is transformed into this form

$$0 = 1 - \frac{z^2}{1 \cdot 2(1-y)} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4(1-y)} - \frac{z^6}{1 \cdot 2 \cdots 6(1-y)} + \text{etc.},$$

one will have the following infinite product for it

$$\left(1 - \frac{zz}{ss}\right)\left(1 - \frac{zz}{(2\pi - s)^2}\right)\left(1 - \frac{zz}{(2\pi + s)^2}\right)\left(1 - \frac{zz}{(4\pi - s)^2}\right)\left(1 - \frac{zz}{(4\pi + s)^2}\right) \text{etc.}$$

COROLLARY 2

§18 Therefore, since in the product the coefficient zz is equal to the sum of the coefficients of zz in the factors, one will have the summation of the following series

$$\frac{1}{2(1-y)} = \frac{1}{ss} + \frac{1}{(2\pi - s)^2} + \frac{1}{(2\pi + s)^2} + \frac{1}{(4\pi - s)^2} + \frac{1}{(4\pi + s)^2} + \text{etc.}$$

COROLLARY 3

§19 Put $s = \frac{m}{n}\pi$ that it is $y = \cos\frac{m}{n}\pi$ and $1 - y = 2\left(\sin\frac{m}{2n}\pi\right)^2$; it will be

$$\frac{\pi\pi}{2nn(1-y)} = \frac{\pi\pi}{4n\left(\sin\frac{m}{2n}\pi\right)^2}$$

$$= \frac{1}{mm} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \frac{1}{(4n-m)^2} + \frac{1}{(4n+m)^2} + \text{etc.},$$

which is identical to the series in § 14, if here one just writes n instead of $2n$.

PROBLEM 3

§20 *To find the roots of z of these infinite equation*

$$t = \frac{z - \frac{z^3}{1\cdot2\cdot3} + \frac{z^5}{1\cdot2\cdot3\cdot4\cdot5} - \frac{z^7}{1\cdot2\cdots7} + \text{etc.}}{1 - \frac{z^2}{1\cdot2} + \frac{z^4}{1\cdot2\cdot3\cdot4} - \frac{z^6}{1\cdot2\cdots6} + \text{etc.}}$$

SOLUTION

If z denotes the arc of the circle whose radius is $= 1$, t will be the tangent of this arc; therefore, if one takes an arc s, whose tangent is t, on this circle, the infinitely many values of z will be the following

$$s, \quad \pi + s, \quad 2\pi + s, \quad 3\pi + s, \quad 4\pi + s, \quad 5\pi + s \quad \text{etc.},$$

$$-\pi + s, \quad -2\pi + s, \quad -3\pi + s, \quad -4\pi + s, \quad -5\pi + s \quad \text{etc.}$$

Q. E. I.

COROLLARY 1

§21 Reduce the propounded equation to this form

$$0 = 1 - \frac{z}{t} - \frac{z^2}{1\cdot2} + \frac{z^3}{1\cdot2\cdot3t} + \frac{z^4}{1\cdot2\cdot3\cdot4} - \frac{z^5}{1\cdot2\cdot3\cdot4\cdot5t} - \text{etc.}$$

and its simple factors will be the following

$$\left(1 - \frac{z}{s}\right)\left(1 - \frac{z}{\pi - s}\right)\left(1 - \frac{z}{\pi + s}\right)\left(1 - \frac{z}{2\pi - s}\right)\left(1 - \frac{z}{2\pi + s}\right)\left(1 - \frac{z}{3\pi - s}\right) \text{ etc.}$$

Corollary 2

§22 Therefore, since the coefficient of z in the equation is equal to the sum of the coefficient of z in the single factors, it will be

$$\frac{1}{t} = \frac{1}{s} - \frac{1}{\pi - s} + \frac{1}{\pi + s} - \frac{1}{2\pi - s} + \frac{1}{2\pi + s} - \frac{1}{3\pi - s} + \text{etc.}$$

Corollary 3

§23 Put $s = \frac{m}{n}\pi$ that it is $t = \tan\frac{m}{n}\pi = \frac{x}{y}$, if it is $x = \sin\frac{m}{n}\pi$ and $y = \cos\frac{m}{n}\pi$, and the summation of the following series will result

$$\frac{\pi}{nt} = \frac{\pi y}{nx} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \text{etc.}$$

Corollary 4

§24 The sum of the squares of the single terms of the series in § 22 is equal to the square of the sum of $\frac{1}{tt}$ diminished by the doubled sum of the products of two terms each, this means -1; therefore, the sum of the squares will be $= \frac{1}{tt} + 1 = \frac{yy}{xx} + 1 = \frac{1}{xx}$. Therefore, as before in § 13, this summation will result

$$\frac{1}{xx} = \frac{1}{ss} + \frac{1}{(\pi - s)^2} + \frac{1}{(\pi + s)^2} + \frac{1}{(2\pi - s)^2} + \frac{1}{(2\pi + s)^2} + \text{etc.}$$

Theorem 1

§25 *While π denotes the half of the circumference of the circle whose radius $= 1$, if it was $x = \sin\frac{m}{n}\pi$, it will be*

$$\frac{\pi(p+qy)}{nx} = \frac{p+q}{m} + \frac{p-q}{n-m} - \frac{p-q}{n+m} - \frac{p+q}{2n-m} + \frac{p+q}{2n+m} + \frac{p-q}{3n-m} - \frac{p-q}{3n+m} - \text{etc.}$$

Proof

If the series found in § 12 is multiplied by p, it will be

$$\frac{\pi p}{nx} = \frac{p}{m} + \frac{p}{n-m} - \frac{p}{n+m} - \frac{p}{2n-m} + \frac{p}{2n+m} + \text{etc.},$$

and if the series in § 23 is multiplied by q, it will be

$$\frac{\pi q y}{n x}=\frac{q}{m}-\frac{q}{n-m}+\frac{q}{n+m}-\frac{q}{2n-m}+\frac{q}{2n+m}-\text{etc.}$$

Add these two series and the propounded series will result, whose sum therefore is $=\frac{\pi(p+qy)}{nx}$. Q. E. D.

COROLLARY 1

§26 Take $p=q$; this summation will result

$$\frac{\pi(1+y)}{2nx}=\frac{1}{m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{4n-m}+\frac{1}{4n+m}-\text{etc.;}$$

but it is $\frac{x}{1+y}=\tan\frac{m}{2n}\pi$, which therefore, having put n instead of $2n$, is identical to the series found in § 23.

COROLLARY 2

§27 If it is $q=-p$, this summation will result

$$\frac{\pi(1-y)}{2nx}=\frac{1}{n-m}-\frac{1}{n+m}+\frac{1}{3n-m}-\frac{1}{3n+m}+\text{etc.;}$$

but it is $\frac{1-y}{x}=\tan\frac{m}{2n}\pi$ and hence

$$\frac{x}{1-y}=\tan\left(\frac{-m}{2n}\pi+\frac{1}{2}\pi\right)=\tan\frac{(n-m)}{2n}\pi,$$

which series having written m instead of $n-m$ again is reduced to the preceding one.

PROBLEM 4

§28 *To find the sum of this series*

$$\frac{1}{m}-\frac{1}{m-n}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\frac{1}{3n-m}-\text{etc.}$$

by means of an integral formula.

SOLUTION

Attribute numerators, which are certain powers of z, whose exponents are equal to the denominators, to the single fractions, and one will have this series

$$\frac{z^m}{m}+\frac{z^{n-m}}{n-m}-\frac{z^{n+m}}{n+m}-\frac{z^{2n-m}}{2n-m}+\frac{z^{2n+m}}{2n+m}+\text{etc.},$$

which having put $z=1$ becomes the propounded one. Put the sum of this series $=s$ and after a differentiation it will be

$$\frac{zds}{dz}=z^m+z^{n-m}-z^{n+m}-z^{2n-m}+z^{2n+m}+z^{3n-m}-\text{etc.},$$

which series is composed of two geometric series, and therefore its sum will be

$$=\frac{z^m}{1+z^n}+\frac{z^{n-m}}{1+z^n}.$$

Therefore, we have

$$\frac{zds}{dz}=\frac{z^m+z^{n-m}}{1+z^n}\quad\text{and}\quad ds=\frac{z^{m-1}+z^{n-m-1}}{1+z^n}dz,$$

as a logical consequence it is

$$s=\int\frac{z^{m-1}+z^{n-m-1}}{1+z^n}dz;$$

therefore, the value of this integral in the case $z=1$ will give the sum of the propounded series. Q. E. I.

COROLLARY 1

§29 Therefore, since the sum of the propounded series is $\frac{\pi}{nx}=\frac{\pi}{n\sin\frac{m}{n}\pi}$, it will be

$$\frac{\pi}{n\sin\frac{m}{n}\pi}=\int\frac{z^{m-1}+z^{n-m-1}}{1+z^n}dz,$$

if one puts $z=1$ after the integration. Therefore, in this case the integral of the formula $\int\frac{z^{m-1}+z^{n-m-1}}{1+z^n}dz$ can be expressed by means of the circle.

COROLLARY 2

§30 Therefore, by substituting definite numbers for m and n one will have the integrations in the case $z = 1$:

If it is $m = 1$, $n = 2$, it will be

$$\frac{\pi}{2} = \int \frac{2dz}{1+zz};$$

if it is $m = 1$, $n = 3$, it will be

$$\frac{2\pi}{3\sqrt{3}} = \int \frac{1+z}{1+z^3}dz = \int \frac{dz}{1-z+zz};$$

if it is $m = 1$, $n = 4$, it will be

$$\frac{\pi}{2\sqrt{2}} = \int \frac{1+zz}{1+z^4}dz;$$

if it is $m = 1$, $n = 6$, it will be

$$\frac{\pi}{3} = \int \frac{1+z^4}{1+z^6}dz;$$

all these, having actually done the integration, are detected to be true.

PROBLEM

§31 *To find the sum of this series*

$$\frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \frac{1}{3n+m} - \text{etc.}$$

by means of an integral formula.

SOLUTION

Combine these fractions with appropriate numerators, as we did it before, and put

$$s = \frac{z^m}{m} - \frac{z^{n-m}}{n-m} + \frac{z^{n+m}}{n+m} - \frac{z^{2n-m}}{2n-m} + \frac{z^{2n+m}}{2n+m} - \text{etc.};$$

and having put $z=1$ the value of s will be the sum of the propounded series. Now take the differentials and this equation will result

$$\frac{zds}{dz}=z^m-z^{n-m}+z^{n+m}-z^{2n-m}+z^{2n+m}-\text{etc.};$$

since the sum of this series can be exhibited, one will have

$$\frac{zds}{dz}=\frac{z^m-z^{n-m}}{1-z^n},$$

whence it is

$$s=\int\frac{z^{m-1}-z^{n-m-1}}{1-z^n}dz.$$

Therefore, the value of this integral formula in the case $z=1$ will give the sum of the propounded series. Q. E. I.

COROLLARY 1

§32 By means of § 23 the sum of the propounded series is $=\frac{\pi}{nt}=\frac{\pi y}{nx}=\frac{\pi\cos\frac{m}{n}\pi}{n\sin\frac{m}{n}\pi}$. Therefore, it will be

$$\frac{\pi\cos\frac{m}{n}\pi}{n\sin\frac{m}{n}\pi}=\int\frac{z^{m-1}-z^{n-m-1}}{1-z^n}dz$$

in the case, in which one puts $z=1$ after the integration.

COROLLARY 2

§33 Therefore, the simpler cases will be as follows:

If it is $m=1$, $n=3$, it will be

$$\frac{\pi}{3\sqrt{3}}=\int\frac{(1-z)dz}{1-z^3}=\int\frac{dz}{1+z+zz};$$

if it is $m=1$, $n=4$, it will be

$$\frac{\pi}{4}=\int\frac{(1-z^2)dz}{1-z^4}=\int\frac{dz}{1+z^2};$$

if it is $m=1$, $n=6$, it will be

$$\frac{\pi}{2\sqrt{3}} = \int \frac{(1-z^4)dz}{1-z^6} = \int \frac{(1+zz)dz}{1+zz+z^4};$$

this formula is easily seen to be true by actual integration.

SCHOLIUM

§34 If in these series we treated up to this point each two terms are combined into one, the following summations will result:

$$\begin{aligned}
\frac{\pi}{n\sin\frac{m}{n}\pi} &= \frac{1}{m} + \frac{2m}{n^2-m^2} - \frac{2m}{4n^2-m^2} + \frac{2m}{9n^2-m^2} - \frac{2m}{16n^2-m^2} + \text{etc.} \\
\frac{\pi\cos\frac{m}{n}\pi}{n\sin\frac{m}{n}\pi} &= \frac{1}{m} - \frac{2m}{n^2-m^2} - \frac{2m}{4n^2-m^2} - \frac{2m}{9n^2-m^2} - \frac{2m}{16n^2-m^2} - \text{etc.}
\end{aligned}$$

Therefore, after some simplifications we will have the following two most remarkable series

$$\begin{aligned}
\frac{\pi}{2mn\sin\frac{m}{n}\pi} - \frac{1}{2mm} &= \frac{1}{n^2-m^2} - \frac{1}{4n^2-m^2} + \frac{1}{9n^2-m^2} - \frac{1}{16n^2-m^2} + \text{etc.} \\
\frac{1}{2mm} - \frac{\pi\cos\frac{m}{n}\pi}{2mn\sin\frac{m}{n}\pi} &= \frac{1}{n^2-m^2} + \frac{1}{4n^2-m^2} + \frac{1}{9n^2-m^2} + \frac{1}{16n^2-m^2} + \text{etc.};
\end{aligned}$$

if they are added to each other, they will give

$$\begin{aligned}
\frac{\pi\left(1-\cos\frac{m}{n}\pi\right)}{4mn\sin\frac{m}{n}\pi} &= \frac{\pi\sin\frac{m}{2n}\pi}{4mn\cos\frac{m}{2n}\pi} \\
&= \frac{1}{n^2-m^2} + \frac{1}{9n^2-m^2} + \frac{1}{25n^2-m^2} + \frac{1}{49n^2-m^2} + \text{etc.}
\end{aligned}$$

And I obtained these series using very different principles several years ago[2].

[2] Euler refers to his paper "De seriebus quibusdam considerationes". This is paper E130 in the Eneström-Index.

THEOREM 2

§35 *The sum of this series of squares*

$$\frac{1}{m^2}-\frac{1}{(n-m)^2}-\frac{1}{(n+m)^2}+\frac{1}{(2n-m)^2}+\frac{1}{(2n+m)^2}-\frac{1}{(3n-m)^2}-\text{etc.}$$

is

$$=\frac{\pi^2\cos\frac{m}{n}\pi}{nn\left(\sin\frac{m}{n}\pi\right)^2}.$$

PROOF

In paragraph § 12 we saw that it is

$$\frac{\pi}{n\sin\frac{m}{n}\pi}=\frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\frac{1}{3n-m}-\frac{1}{3n+m}-\text{etc.};$$

since this equality must always hold, whatever m is, the differentials must also be identical. Therefore, let us differentiate so the series as the sum with respect to m and also the resulting differentials will be equal. Having divided by dm on both sides one will have

$$\frac{-\pi\cos\frac{m}{n}\pi}{nn\left(\sin\frac{m}{n}\pi\right)}=-\frac{1}{m^2}+\frac{1}{(n-m)^2}+\frac{1}{(n+m)^2}-\frac{1}{(2n-m)^2}-\frac{1}{(2n+m)^2}+\text{etc.}$$

Change the sings and one will have the sum of the propounded series of squares. Q. E. D.

COROLLARY 1

§36 Let us expand some simpler cases and let $m=1$, $n=2$; it will be

$$\sin\frac{m}{n}\pi=1\quad\text{and}\quad\cos\frac{m}{n}\pi=0,$$

whence it is

$$0=1-1-\frac{1}{9}+\frac{1}{9}+\frac{1}{25}-\frac{1}{25}-\frac{1}{49}+\text{etc.}$$

Let $m = 1$, $n = 3$; it will be

$$\sin\frac{m}{n}\pi = \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos\frac{m}{n}\pi = \frac{1}{2},$$

whence

$$\frac{2\pi\pi}{27} = 1 - \frac{1}{4} - \frac{1}{16} + \frac{1}{25} + \frac{1}{49} - \frac{1}{64} - \frac{1}{100} + \text{etc.}$$

Let $m = 1$, $n = 4$; it will be

$$\sin\frac{m}{n}\pi = \cos\frac{m}{n}\pi = \frac{1}{\sqrt{2}},$$

whence it is

$$\frac{\pi\pi}{8\sqrt{2}} = 1 - \frac{1}{9} - \frac{1}{25} + \frac{1}{49} + \frac{1}{81} - \frac{1}{121} - \frac{1}{169} + \text{etc.}$$

COROLLARY 2

§37 Let us multiply the series

$$\frac{2\pi\pi}{27} = 1 - \frac{1}{4} - \frac{1}{16} + \frac{1}{25} + \frac{1}{49} - \frac{1}{64} - \frac{1}{100} + \text{etc.},$$

in which the squares divisible by three are missing, by

$$\frac{9}{8} = 1 + \frac{1}{9} + \frac{1}{81} + \frac{1}{729} + \text{etc.},$$

that all squares occur, and it will be

$$\frac{\pi\pi}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \frac{1}{49} - \text{etc.},$$

which I proved already some time ago[3].

[3] Euler refers to E61 again.

COROLLARY 3

§38 Because from § 14 it is

$$\frac{\pi\pi}{nnxx} = \frac{\pi\pi}{nn\left(\sin\frac{m}{n}\pi\right)^2} = \frac{1}{m^2} + \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \text{etc.},$$

having added these series it will be

$$\frac{\pi\pi\left(1+\cos\frac{m}{n}\pi\right)}{2nn\left(\sin\frac{m}{n}\pi\right)^2} = \frac{1}{m^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \frac{1}{(4n-m)^2} + \frac{1}{(4n+m)^2} + \text{etc.},$$

which series by writing n instead of $2n$ is reduced to the first one; for, it is

$$\frac{1+\cos\frac{m}{n}\pi}{\left(\sin\frac{m}{n}\pi\right)^2} = \frac{1}{2\left(\sin\frac{m}{2n}\pi\right)^2}.$$

SCHOLIUM

§39 Therefore, the summation proved in this proposition could have been deduced directly from the summation of the series given in § 14. For, since it is

$$\frac{\pi\pi}{nn\left(\sin\frac{m}{n}\pi\right)^2} = \frac{1}{m^2} + \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \text{etc.},$$

by writing $2n$ instead n it will be

$$\frac{\pi\pi}{4nn\left(\sin\frac{m}{2n}\pi\right)^2} = \frac{\pi\pi\left(1+\cos\frac{m}{n}\pi\right)}{2nn\left(\sin\frac{m}{n}\pi\right)^2}$$
$$= \frac{1}{m^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \frac{1}{(4n-m)^2} + \frac{1}{(4n+m)^2} + \text{etc.};$$

if the first series is subtracted from the double of this series, the propounded series will remain

$$\frac{\pi\pi\cos\frac{m}{n}\pi}{nn\left(\sin\frac{m}{n}\pi\right)^2} = \frac{1}{m^2} - \frac{1}{(n-m)^2} - \frac{1}{(n+m)^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} - \text{etc.};$$

but in like manner it is possible to deduce the series exhibited in § 12 from the series in § 23, which because of the alternating signs seems to be highly regular. For, since it is

$$\frac{\pi\cos\frac{m}{n}\pi}{n\sin\frac{m}{n}\pi} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \text{etc.},$$

if one writes n instead of $2n$, it will be

$$\frac{\pi\cos\frac{m}{2n}\pi}{2n\sin\frac{m}{2n}\pi} = \frac{\pi\left(1+\cos\frac{m}{n}\pi\right)}{2n\sin\frac{m}{n}\pi} = \frac{1}{m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{4n-m} + \frac{1}{4n+m} - \text{etc.}$$

Subtract that series from the double of this last series and it will be

$$\frac{\pi}{n\sin\frac{m}{n}\pi} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \text{etc.},$$

which is the series found in § 12.

But in like manner the integral formulas, which were found for these sums, are reduced to each other. For, because it is (§ 32)

$$\frac{\pi\cos\frac{m}{n}\pi}{n\sin\frac{m}{n}\pi} = \int \frac{z^{m-1} - z^{n-m-1}}{1-z^n}dz,$$

it will also be

$$\frac{\pi\cos\frac{m}{2n}\pi}{2n\sin\frac{m}{2n}} = \frac{\pi\left(1+\cos\frac{m}{n}\pi\right)}{2n\sin\frac{m}{n}\pi} = \int \frac{z^{m-1} - z^{2n-m-1}}{1-z^{2n}}dz;$$

subtract the first from the double of this integral; it will be

$$\frac{\pi}{n\sin\frac{m}{n}\pi} = \int \frac{2z^{m-1} - 2z^{2n-m-1}}{1-z^{2n}}dz - \int \frac{z^{m-1} - z^{n-m-1}}{1-z^n}dz$$
$$= \int \frac{z^{m-1} - z^{n+m-1} + z^{n-m-1} - z^{2n-m-1}}{1-z^{2n}}dz = \int \frac{z^{m-1} + z^{n-m-1}}{1+z^n}dz,$$

which is the integration found in § 29. From these considerations it is perspicuous that everything, what was found up to this point, could have been found from this summation

$$\frac{\pi\cos\frac{m}{n}\pi}{n\sin\frac{m}{n}\pi} = \int\frac{z^{m-1}-z^{n-m-1}}{1-z^n}dz$$

$$= \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \frac{1}{3n+m} - \text{etc.}$$

By differentiating this series this one results

$$\frac{\pi\pi}{nn\left(\sin\frac{m}{n}\pi\right)^2} = \frac{1}{m^2} + \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \text{etc.},$$

which was already found in § 14.

Problem 5

§40 *To find the differentials of first, second and all following higher orders of this quantity*

$$\frac{\pi\cos\frac{m}{n}\pi}{n\sin\frac{m}{n}\pi}$$

having considered m as a variable.

Solution

For the sake of brevity let us put

$$\sin\frac{m}{n}\pi = x \quad \text{and} \quad \cos\frac{m}{n}\pi = y;$$

at first it will be $y = \sqrt{1-xx}$; but then it will be

$$dx = \frac{\pi dm}{n}y = \frac{\pi y}{n}dm \quad \text{and} \quad dy = -\frac{\pi x}{n}dm.$$

Additionally, call the propounded quantity, whose differential is in question,

$$\frac{\pi\cos\frac{m}{n}\pi}{n\sin\frac{m}{n}\pi} = V;$$

it will be $V = \frac{\pi y}{nx}$. Therefore, it will hence be

$$dV = \frac{\pi(xdy - ydx)}{nxx} = \frac{-\pi\pi dm}{nnxx}$$

because of $xx + yy = 1$ and hence

$$\frac{dV}{dm} = \frac{-\pi\pi}{nn} \cdot \frac{1}{xx};$$

further, differentiate this expression and it will be

$$\frac{ddV}{dm^2} = +\frac{\pi\pi}{nn} \cdot \frac{2dx}{x^3} = \frac{2\pi^3}{n^3} \cdot \frac{ydm}{x^3}$$

and hence

$$\frac{d^2V}{dm^2} = \frac{\pi^3}{n^3} \cdot \frac{2y}{x^3}.$$

If the following differentials are computed in the same way, they will be as follows:

$$\begin{aligned}
V &= +\frac{\pi}{nx} \cdot y, \\
\frac{dV}{dm} &= -\frac{\pi^2}{n^2x^2} \cdot 1, \\
\frac{ddV}{dm^2} &= +\frac{\pi^3}{n^3x^3} \cdot 2y, \\
\frac{d^3V}{dm^3} &= -\frac{\pi^4}{n^4x^4}(4yy + 2), \\
\frac{d^4V}{dm^4} &= +\frac{\pi^5}{n^5x^5}(8y^3 + 16y), \\
\frac{d^5V}{dm^5} &= -\frac{\pi^6}{n^6x^6}(16y^4 + 88y^2 + 16), \\
\frac{d^6V}{dm^6} &= +\frac{\pi^7}{n^7x^7}(32y^5 + 416y^3 + 272y), \\
\frac{d^7V}{dm^7} &= -\frac{\pi^8}{n^8x^8}(64y^6 + 1824y^4 + 2880y^2 + 272) \\
&+ \text{etc.}
\end{aligned}$$

The law of progression is, that, if it was

$$\frac{d^\nu V}{dm^\nu} = \pm \frac{\pi^{\nu+1}}{n^{\nu+1}x^{\nu+1}} = (\alpha y^{\nu-1} + \beta y^{\nu-3} + \gamma y^{\nu-5} + \delta y^{\nu-7} + \varepsilon y^{\nu-9} + \text{etc.}),$$

the following order of differentiation will be

$$\frac{d^{\nu+1} V}{dm^{\nu+1}} = \mp \frac{\pi^{\nu+2}}{n^{\nu+2}x^{\nu+2}} \begin{Bmatrix} 2\alpha y^\nu + (4\beta + (\nu-1)\alpha)y^{\nu-2} + (6\gamma + (\nu-3)\beta)y^{\nu-4} \\ +(8\delta + (\nu-5)\gamma)y^{\nu-6} + (10\varepsilon + (\nu-7)\delta)y^{\nu-8} + \text{etc.} \end{Bmatrix}$$

Therefore, the differentials of each order will be determined from the preceding ones. Q. E. I.

PROBLEM 6

§41 *To find the sum of this series*

$$\frac{1}{m^\nu} + \frac{1}{(m-n)^\nu} + \frac{1}{(m+n)^\nu} + \frac{1}{(m-2n)^\nu} + \frac{1}{(m+2n)^\nu} + \frac{1}{(m-3n)^\nu} + \text{etc.}$$

having raised the single terms of the series found in § 23 to an arbitrary power.

SOLUTION

If we put $\sin \frac{m}{n}\pi = x$, $\cos \frac{m}{n}\pi = y$ and $\frac{\pi y}{nx} = V$, from § 23 it will be

$$V = \frac{1}{m} + \frac{1}{m-n} + \frac{1}{m+n} + \frac{1}{m-2n} + \frac{1}{m+2n} + \frac{1}{m-3n} + \text{etc.}$$

If one now differentiates with respect to m, the following summations will result:

$$
\begin{aligned}
- \quad \frac{dV}{1dm} &= \frac{1}{m^2} + \frac{1}{(m-n)^2} + \frac{1}{(m+n)^2} + \frac{1}{(m-2n)^2} + \frac{1}{(m+2n)^2} + \text{etc.},\\
+ \quad \frac{ddV}{1\cdot 2dm^2} &= \frac{1}{m^3} + \frac{1}{(m-n)^3} + \frac{1}{(m+n)^3} + \frac{1}{(m-2n)^3} + \frac{1}{(m+2n)^3} + \text{etc.},\\
- \quad \frac{d^3V}{1\cdot 2\cdot 3dm^3} &= \frac{1}{m^4} + \frac{1}{(m-n)^4} + \frac{1}{(m+n)^4} + \frac{1}{(m-2n)^4} + \frac{1}{(m+2n)^4} + \text{etc.},\\
+ \frac{d^4V}{1\cdot 2\cdot 3\cdot 4dm^4} &= \frac{1}{m^5} + \frac{1}{(m-n)^5} + \frac{1}{(m+n)^5} + \frac{1}{(m-2n)^5} + \frac{1}{(m+2n)^5} + \text{etc.}\\
&\text{etc.}
\end{aligned}
$$

Therefore, the sum of the propounded series of indefinite degree

$$\frac{1}{m^\nu} + \frac{1}{(m-n)^\nu} + \frac{1}{(m+n)^\nu} + \frac{1}{(m-2n)^\nu} + \frac{1}{(m+2n)^\nu} + \frac{1}{(m-3n)^\nu} + \text{etc.}$$

will be

$$= \frac{\pm d^{\nu-1}V}{1\cdot 2\cdot 3\cdots(\nu-1)dm^{\nu-1}}.$$

But in the preceding problem we exhibited the value of $\frac{d^{\nu-1}V}{dm^{\nu-1}}$; therefore, it will also be possible to define the sum of these series of powers. Q. E. I.

PROBLEM 7

§42 *To exhibit the sine of the arbitrary angle $\frac{m}{n}\pi$ by means of an infinite product.*

SOLUTION

Because it is

$$\frac{\pi\cos\frac{m}{n}\pi}{n\sin\frac{m}{n}\pi} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \text{etc.},$$

treat m as the variable and multiply both sides by dm; it will be

$$\frac{\pi dm}{n}\cos\frac{m}{n}\pi = d.\sin\frac{m}{n}\pi$$

and therefore it will be

$$\frac{\pi dm\cos\frac{m}{n}\pi}{n\sin\frac{m}{n}\pi}=\frac{\sin\frac{m}{n}\pi}{\sin\frac{m}{n}\pi}=\frac{dm}{m}-\frac{dm}{n-m}+\frac{dm}{n+m}-\frac{dm}{n+m}-\frac{dm}{2n-m}+\frac{dm}{2n+m}-\text{etc.},$$

whence having integrated both sides it will be

$$\log\sin\frac{m}{n}\pi=\log m+\log(n-m)+\log(n+m)+\log(2n-m)+\log(2n+m)+\text{etc.}+C.$$

The constant C must be of such a nature, that for $m=\frac{1}{2}n$ the logarithm of the sine becomes $=0$, in which case we have a sinus totus[4]. Therefore, having determined the constant this way it will be

$$\log\sin\frac{m}{n}\pi=\log\frac{2m}{n}+\log\frac{2n-2m}{n}+\log\frac{2n+2m}{3n}+\log\frac{4n-2m}{3n}+\log\frac{4n+2m}{5n}+\text{etc.}$$

Hence, if we go back to numbers, we will have

$$\sin\frac{m}{n}\pi=\frac{2m}{n}\cdot\frac{2n-2m}{n}\cdot\frac{2n+2m}{3n}\cdot\frac{4n-2m}{3n}\cdot\frac{4n+2m}{5n}\cdot\text{etc.}$$

Or if we multiply each two factors by each other, it will be

$$\sin\frac{m}{n}\pi=\frac{2n}{m}\cdot\frac{4nn-4mm}{3nn}\cdot\frac{16nn-4mm}{15nn}\cdot\frac{36nn-4mm}{35nn}\cdot\text{etc.}$$

Q. E. I.

Corollary 1

§43 If we write m instead of $2m$, we will have

$$\sin\frac{m}{2n}\pi=\frac{m}{n}\cdot\frac{4nn-mm}{4nn-nn}\cdot\frac{16nn-mm}{16nn-nn}\cdot\frac{26nn-mm}{36nn-nn}\cdot\text{etc.}$$

[4]By this Euler means $\sin\frac{\pi}{2}$

COROLLARY 2

§44 Since it is $\sin\frac{m}{2n}\pi = \cos\frac{(n-m)}{2n}\pi$, if we write $n-m$ instead of m, from the found series it will be

$$\cos\frac{m}{2n}\pi = \frac{n-m}{n}\cdot\frac{n+m}{n}\cdot\frac{3n-m}{3n}\cdot\frac{3n+m}{3n}\cdot\text{etc.}$$

or

$$\cos\frac{m}{2n}\pi = \frac{nn-mm}{nn}\cdot\frac{9nn-mm}{9nn}\cdot\frac{25nn-mm}{25nn}\cdot\text{etc.}$$

COROLLARY 3

§45 Since it is $\sin\frac{m}{n}\pi = 2\sin\frac{m}{2n}\pi\cdot\cos\frac{m}{2n}\pi$, if we divide $\sin\frac{m}{n}\pi$ by $2\sin\frac{m}{2n}\pi$, we will have

$$\cos\frac{m}{2n}\pi = \frac{2n-2n}{2n-m}\cdot\frac{2n+2m}{2n+m}\cdot\frac{4n-2m}{4n-m}\cdot\frac{4n+2m}{4n+m}\cdot\text{etc.}$$

and having divided $\sin\frac{m}{n}\pi$ by $2\cos\frac{m}{2n}\pi$ we will have

$$\sin\frac{m}{2n}\pi = \frac{m}{n-m}\cdot\frac{2n-2m}{n+m}\cdot\frac{2n+2m}{3n-n}\cdot\frac{4n-2m}{3n+m}\cdot\frac{4n+2m}{5n-m}\cdot\text{etc.}$$

COROLLARY 4

§46 These two expressions of sines and cosines, having set them equal to each other, will give

$$1 = \frac{nn}{nn-mm}\cdot\frac{4nn-4mm}{4nn-mm}\cdot\frac{9nn}{9nn-mm}\cdot\frac{16nn-4mm}{16nn-mm}\cdot\frac{25nn}{25nn-mm}\cdot\text{etc.}$$

COROLLARY 5

§47 If n is assumed to be infinitely large or m infinitely small, it will be $\sin\frac{m}{2n}\pi = \frac{m}{2n}\pi$ and in this case from each of the two series the same value of π, which was given by Wallis, will result

$$\pi = 2\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\frac{8\cdot 8}{7\cdot 9}\cdot\frac{10\cdot 10}{9\cdot 11}\cdot\text{etc.}$$

LEMMA 4

§48 *The value of this infinite product*

$$\frac{a(c+b)(a+k)(c+b+k)(a+2k)(c+b+2k)\text{ etc.}}{b(c+a)(b+k)(c+a+k)(b+2k)(c+a+2k)\text{ etc.}}$$

is

$$=\frac{\int z^{c-1}dz(1-z^k)^{\frac{b-k}{k}}}{\int z^{c-1}dz(1-z^k)^{\frac{a-k}{k}}},$$

if after each of the integrations one puts $z=1$.

PROBLEM 8

§49 *To express the sine of the angle $\frac{m}{2n}\pi$ by means of integral formulas.*

SOLUTION

Since it is

$$\sin\frac{m}{2n}\pi=\frac{m}{n}\cdot\frac{2n-m}{n}\cdot\frac{2n+m}{3n}\cdot\frac{4n-m}{3n}\cdot\text{etc.}$$

using the results of § 43 compare this infinite product to the preceding lemma and it will be $a=m$, $b=n$, $k=2n$ and $c+m=n$ or $c+n=2n-m$; both of these conditions give $c=n-m$. Therefore, it will hence be

$$\sin\frac{m}{2n}\pi=\frac{\int z^{n-m-1}dz(1-z^{2n})^{-\frac{1}{2}}}{\int z^{n-m-1}dz(1-z^{2n})^{\frac{-2n+m}{2n}}},$$

if after both integrations done in such a way that they vanish having put $z=0$ one sets $z=1$.

But because by § 45 it also is

$$2\sin\frac{m}{2n}\pi=\frac{2m}{n-m}\cdot\frac{2n-2m}{n+m}\cdot\frac{2n+2m}{3n-m}\cdot\frac{4n-2m}{3n+m}\cdot\text{etc.},$$

after the comparison to the lemma it will be $a=2m$, $b=n-m$, $c=n-m$ and $k=2n$, whence one will obtain

$$2\sin\frac{m}{2n}\pi = \frac{\int z^{n-m-1}dz(1-z^{2n})^{\frac{-n-m}{2n}}}{\int z^{n-m-1}dz(1-z^{2n})^{\frac{m-n}{n}}},$$

if one puts $z=1$ after the integration. Q. E. I.

Corollary 1

§50 Therefore, we obtain the following comparisons of different integral formulas:

$$\sin\frac{m}{2n}\pi\cdot\int\frac{z^{n-m-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}} = \int\frac{z^{n-m-1}dz}{\sqrt{1-z^{2n}}}$$

and

$$2\sin\frac{m}{2n}\pi\cdot\int\frac{z^{n-m-1}dz}{(1-z^{2n})^{\frac{n-m}{n}}} = \int\frac{z^{n-m-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}}.$$

Corollary 2

§51 But then not using the sine one has this relation

$$2\int\frac{z^{n-m-1}dz}{(1-z^{2n})^{\frac{n-m}{n}}} : \int\frac{z^{n-m-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}} = \int\frac{z^{n-m-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}} : \int\frac{z^{n-m-1}dz}{\sqrt{1-z^{2n}}}.$$

Corollary 3

§52 Let us put that it is $m=1$ and $n=1$; it will be $\sin\frac{m}{2n}\pi = 1$ and the relations will be as follows

$$\int\frac{dz}{z\sqrt{1-zz}} = \int\frac{dz}{z\sqrt{1-zz}} \quad \text{and} \quad 2\int\frac{dz}{z} = \int\frac{dz}{z(1-zz)};$$

in the second equation two infinite areas under a hyperbola are compared to each other.

Corollary 4

§53 Let us put that it is $m = 2$ and $n = 3$; it will be $\sin \frac{m}{2n}\pi = \frac{\sqrt{3}}{2}$, whence the following comparisons result

$$\frac{\sqrt{3}}{2}\int \frac{dz}{(1-z^6)^{\frac{2}{3}}} = \int \frac{dz}{\sqrt{1-z^6}} \quad \text{and} \quad \sqrt{3}\int \frac{dz}{(1-z^6)^{\frac{1}{3}}} = \int \frac{dz}{(1-z^6)^{\frac{5}{6}}};$$

from these this proportion results

$$\frac{1}{2}\int \frac{dz}{(1-z^6)^{\frac{2}{3}}} : \int \frac{dz}{(1-z^6)^{\frac{1}{3}}} = \int \frac{dz}{(1-z^6)^{\frac{1}{3}}} : \int \frac{dz}{(1-z^6)^{\frac{5}{6}}}.$$

Corollary 5

§54 Let $m = 1$, $n = 2$ that it is $\sin \frac{m}{2n}\pi = \frac{1}{\sqrt{2}}$; it will be

$$\frac{1}{\sqrt{2}}\int \frac{dz}{(1-z^4)^{\frac{3}{4}}} = \int \frac{dz}{(1-z^4)^{\frac{1}{2}}} \quad \text{and} \quad \sqrt{2}\int \frac{dz}{(1-z^4)^{\frac{1}{2}}} = \int \frac{dz}{(1-z^4)^{\frac{3}{4}}},$$

which two equations are identical.

Corollary 6

§55 Let $m = 1$, $n = 3$ that it is $\sin \frac{m}{2n}\pi = \frac{1}{2}$; it will be

$$\frac{1}{2}\int \frac{zdz}{(1-z^6)^{\frac{5}{6}}} = \int \frac{zdz}{(1-z^6)^{\frac{1}{2}}} \quad \text{and} \quad \int \frac{zdz}{(1-z^6)^{\frac{2}{3}}} = \int \frac{zdz}{(1-z^6)^{\frac{2}{3}}},$$

the second equation is just the identical equation, but the first gives

$$\int \frac{zdz}{(1-z^6)^{\frac{5}{6}}} = 2\int \frac{zdz}{\sqrt{1-z^6}}$$

having put $z = 1$ after the integration, which conditions is to be understood to be always added.

Problem 9

§56 *To express the infinite expressions we found for the cosine of the angle $\frac{m}{2n}\pi$ to integral formulas.*

SOLUTION

First, in § 44 we found that it is

$$\cos\frac{m}{2n}\pi = \frac{n-m}{n}\cdot\frac{n+m}{n}\cdot\frac{3n-m}{3n}\cdot\frac{3n+m}{3n}\cdot\text{etc.},$$

which expression having compared it to the lemma in § 48 gives $a = n - m$, $b = n$, $c = m$ and $k = 2n$, having substituted which values it then results

$$\cos\frac{m}{2n}\pi = \frac{\int z^{m-1}dz(1-z^{2n})^{-\frac{1}{2}}}{\int z^{m-1}dz(1-z^{2n})^{\frac{-n-m}{2n}}}.$$

Further, in § 45 we saw that it is

$$\cos\frac{m}{2n}\pi = \frac{2n-2m}{2n-m}\cdot\frac{2n+2m}{2n+m}\cdot\frac{4n-2m}{4n-m}\cdot\frac{4n+2m}{4n+m}\cdot\text{etc.};$$

having compared this expression to the lemma one will find $a = 2n - 2m$, $b = 2n - m$, $c = 3m$ and $k = 2n$, whence it will be

$$\cos\frac{m}{2n}\pi = \frac{\int z^{3m-1}dz(1-z^{2n})^{-\frac{m}{2n}}}{\int z^{3m-1}dz(1-z^{2n})^{-\frac{m}{n}}},$$

if one puts $z = 1$ after the integration. Q. E. I.

COROLLARY 1

§57 Hence one can express the sine of the angle $\frac{m}{2n}\pi$ again by putting $n - m$ instead of m; the first expression certainly gives the one we already found, but from the second this one results

$$\sin\frac{m}{2n}\pi = \frac{\int z^{3n-3m-1}dz(1-z^{2n})^{\frac{-n+m}{2n}}}{\int z^{3n-3m-1}dz(1-z^{2n})^{\frac{-n+m}{n}}}$$

COROLLARY 2

§58 As we had three expressions for the sine, so we will also have a third expression for the cosine from the second expression of the sine [§ 49], which will give

$$2\cos\frac{m}{2n}\pi = \frac{\int z^{m-1}dz(1-z^{2n})^{\frac{-2n+m}{2n}}}{\int z^{m-1}dz(1-z^{2n})^{\frac{-m}{n}}}.$$

COROLLARY 3

§59 Therefore, one will hence be able to compare innumerable pairs of integrals in the case, in which $z = 1$, and these comparisons will depend on the multisection of the angle.

PROBLEM 10

§60 *To find integral expressions which exhibit the tangent of the angle $\frac{m}{2n}\pi$ in the case $z = 1$.*

SOLUTION

Since the tangent is the quotient resulting from the division of the sine by the cosine, from § 43 and 44 it will be

$$\tan\frac{m}{2n}\pi = \frac{m}{n-m}\cdot\frac{2n-m}{n+m}\cdot\frac{2n+m}{3n-m}\cdot\frac{4n-m}{3n+m}\cdot\text{etc.};$$

compare this expression to the lemma given in § 48 and it will be $a = m$, $b = n - m$, $c = n$ and $k = 2n$, whence this equation will result

$$\tan\frac{m}{2n}\pi = \frac{\int z^{n-1}dz(1-z^{2n})^{\frac{-n-m}{2n}}}{\int z^{n-1}dz(1-z^{2n})^{\frac{m-2n}{2n}}}$$

having put $z = 1$ after each of both integrations. Further, from § 45 one finds this expression for the tangent

$$\tan\frac{m}{2n}\pi = \frac{m}{n-m}\cdot\frac{2n-m}{n+m}\cdot\frac{2n+m}{3n-m}\cdot\frac{4n-m}{3n+m}\cdot\text{etc.},$$

from which one finds the same integral expression as before. Q. E. I.

COROLLARY 1

§61 Let us put $m = 2$ and $n = 3$; it will be $\tan\frac{m}{2n}\pi = \sqrt{3}$ and hence

$$\sqrt{3}\int\frac{zzdz}{(1-z^6)^{\frac{2}{3}}} = \int\frac{zzdz}{(1-z^6)^{\frac{5}{6}}};$$

if we put $z^3 = v$, it will be $zzdz = \frac{1}{3}dv$ and therefore

$$\int \frac{dv\sqrt{3}}{(1-v^2)^{\frac{2}{3}}} = \int \frac{dv}{(1-v^2)^{\frac{5}{6}}}.$$

COROLLARY 2

§62 If we in general put $z^n = v$, one will have

$$\tan \frac{m}{2n}\pi = \frac{\int dv(1-vv)^{\frac{-n-m}{2n}}}{\int dv(1-vv)^{\frac{m-2n}{2n}}}$$

or

$$\tan \frac{m}{2n}\pi \cdot \int \frac{dv}{(1-vv)^{\frac{2n-m}{2n}}} = \int \frac{dv}{(1-vv)^{\frac{n+m}{2n}}}.$$

COROLLARY 3

§63 Let $m = 1$, $n = 2$; it will be $\tan \frac{m}{2n}\pi = 1$ and hence

$$\int \frac{dv}{(1-vv)^{\frac{3}{4}}} = \int \frac{dv}{(1-vv)^{\frac{3}{4}}},$$

which identical equation reveals that the calculation was correct.

SCHOLIUM

§64 It is possible to find many relations of this kind, if one recalls the theorems on the comparisons of integrals formulas I proved in another paper[5], from which I will present some of them as lemmas here.

LEMMA 5

§65 *If one puts $z = 1$ after the integration everywhere, it will be*

$$\int \frac{z^{a-1}dz}{(1-z^b)^{1-c}} \cdot \int \frac{z^{a+bc-1}dz}{(1-z^b)^{1-\gamma}} = \int \frac{z^{a-1}dz}{(1-z^b)^{1-\gamma}} \cdot \int \frac{z^{a+b\gamma-1}dz}{(1-z^b)^{1-c}}.$$

[5] Euler refers to his paper "De productis ex infinitis factoribus ortis". This is E122 in the Eneström-Index.

Lemma 6

§66 *If one puts $z=1$ after the integrations, it will be*

$$\frac{b+1}{c+1}=\frac{\int z^{b(\frac{1}{2}-k)-1}dz(1-z^b)^c\cdot\int z^{b(\frac{3}{2}+c-k)-1}dz(1-z^b)^{-\frac{1}{2}+k}}{\int z^{c(\frac{1}{2}-k)-1}dz(1-z^c)^b\cdot\int z^{c(\frac{3}{2}+b+k)-1}dz(1-z^c)^{-\frac{1}{2}-k}}.$$

Lemma 7

§67 *If one puts $z=1$ after the integrations, it will be*

$$\frac{c}{a}=\frac{\int z^{a-1}dz(1-z^b)^{-\frac{1}{2}+k}\cdot\int z^{a+(\frac{1}{2}+k)b-1}dz(1-z^b)^{-\frac{1}{2}-k}}{\int z^{c-1}dz(1-z^b)^{-\frac{1}{2}-k}\cdot\int z^{c+(\frac{1}{2}-k)b-1}dz(1-z^b)^{-\frac{1}{2}+k}}.$$

Lemma 8

§68 *If one puts $z=1$ after the integration, it will be*

$$\frac{(a+1)(a-k+1)}{(c+1)(c+k+1)}=\frac{\int z^{b(1+k)-1}dz(1-z^b)^c\cdot\int z^{b(1-k)-1}dz(1-z^b)^{c+k}}{\int z^{b(1-k)-1}dz(1-z^b)^a\cdot\int z^{b(1+k)-1}dz(1-z^b)^{a-k}}.$$

Theorem 3

§69 *If one puts $z=1$ after the integrations, it will be*

$$\cos\frac{m}{2n}\pi=\frac{\int\frac{z^{m-1}dz}{(1-z^{2n})^{\frac{1}{2}}}\cdot\int\frac{z^{n-1}dz}{(1-z^{2n})^{1-c}}}{\int\frac{z^{m-1}dz}{(1-z^{2n})^{1-c}}\cdot\int\frac{z^{m+2nc-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}}}.$$

Proof

For, if we put $a=m$, $b=2n$ and $\gamma=\frac{n-m}{2n}$, it is

$$\int\frac{z^{a-1}dz}{(1-z^b)^{1-\gamma}}=\int\frac{z^{m-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}}.$$

But by § 56 it is

$$\int \frac{z^{m-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}} = \frac{1}{\cos\frac{m}{2n}\pi} \int \frac{z^{m-1}dz}{(1-z^{2n})^{\frac{1}{2}}},$$

which value, having substituted it in the lemma, will give the equality which is to be demonstrated.

COROLLARY 1

§70 This equality contains the indefinite index c which can be chosen arbitrarily; therefore, let $c = \frac{1}{2}$, and since the numerator and the denominator have a common factor, it will be

$$\cos\frac{m}{2n}\pi \cdot \int \frac{z^{n+m-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}} = \int \frac{z^{n-1}dz}{\sqrt{1-z^{2n}}}.$$

COROLLARY 2

§71 If we put $\int \frac{z^{n-1}dz}{\sqrt{1-z^{2n}}}$ in the formula, it goes over into $\frac{1}{n}\int \frac{dv}{\sqrt{1-vv}}$, whose integral having put $z = 1$ or $v = 1$ will be $\frac{\pi}{2n}$. Therefore, it will be

$$\int \frac{z^{n+m-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}} = \frac{\pi}{2n\cos\frac{m}{2n}\pi}$$

having put $z = 1$.

COROLLARY 3

§72 If we put $z = \frac{u}{(1+u^{2n})^{\frac{1}{2n}}}$, that we introduce u instead of the variable z, it will be $z = 0$, if $u = 0$, but it is $z = 1$, if it is $u = \infty$. Therefore, after the substitution it will be

$$\int \frac{u^{n+m-1}du}{1+u^{2n}} = \frac{\pi}{2n\cos\frac{m}{2n}\pi}$$

having put $u = \infty$ after the integration.

COROLLARY 4

§73 If we put $2n$ instead of n in § 29, it will be

$$\frac{\pi}{2n\sin\frac{m}{2n}\pi} = \int \frac{z^{m-1} + z^{2n-m-1}}{1+z^{2n}}dz,$$

if it is $z = 1$ after the integration. Therefore, if one writes $n - m$ for m, it will be

$$\frac{\pi}{2n\cos\frac{m}{2n}\pi} = \int \frac{z^{n-m-1} + z^{n+m-1}}{1+z^{2n}}dz$$

having put $z = 1$ after the integration, which integral is hence equal to this one $\int \frac{u^{n+m-1}du}{1+u^{2n}}$, if one puts $u = \infty$.

THEOREM 4

§74 *If one puts $z = 1$ after the integrations, it will be*

$$2\cos\frac{m}{2n}\pi \cdot \int \frac{z^{2m-1}dz}{(1-z^{2n})^{\frac{m}{n}}} = \int \frac{z^{2n-m-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}}.$$

PROOF

In § 58 we obtained this expression for the cosine

$$2\cos\frac{m}{2n}\pi = \frac{\int z^{m-1}dz(1-z^{2n})^{\frac{-2n+m}{2n}}}{\int z^{m-1}dz(1-z^{2n})^{-\frac{m}{n}}}.$$

If we now put $a = m$, $b = 2n$, $c = \frac{m}{2n}$ and $\gamma = \frac{n-m}{n}$ in Lemma 5, the two integral formulas of the lemma are transformed into these, which express $2\cos\frac{m}{2n}\pi$; if one writes $2\cos\frac{m}{2n}\pi$ instead of them, this equation will result

$$2\cos\frac{m}{2n}\pi \cdot \int \frac{z^{2m-1}dz}{(1-z^{2n})^{\frac{m}{n}}} = \int \frac{z^{2n-m-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}}.$$

Q. E. D.

COROLLARY 1

§75 Hence, if one puts $b = 2n$, $c = \frac{-m}{n}$ and $k = \frac{n-2m}{2n}$ in Lemma 6, the formula $\int z^{b(\frac{1}{2}-k)-1}dz(1-z^b)^c$ goes over into this one $\int \frac{z^{2m-1}dz}{(1-z^{2n})^{\frac{m}{n}}}$; if we write

$$\frac{1}{2\cos\frac{m}{2n}\pi}\int\frac{z^{2n-m-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}}$$

instead of this formula and put $b = 0$, we will obtain this reduction

$$2\cos\frac{m}{2n}\pi = \frac{\int\frac{z^{2n-m-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}}}{\int\frac{z^{2n-m-1}dz}{(1-z^{2n})^{\frac{n-m}{2n}}}} \quad \text{or} \quad \int\frac{z^{2m-1}dz}{(1-z^{2n})^{\frac{m}{n}}} = \int\frac{z^{2n-2m-1}dz}{(1-z^{2n})^{\frac{n-m}{n}}}.$$

COROLLARY 2

§76 If we put $m = 2$ and $n = 3$, it will be $\cos\frac{m}{2n}\pi = \frac{1}{2}$, whence the equation of the theorem will give us

$$\int\frac{z^3dz}{(1-z^6)^{\frac{2}{3}}} = \int\frac{z^3dz}{(1-z^6)^{\frac{2}{3}}},$$

but the corollary of the preceding corollary will give us

$$\int\frac{zdz}{(1-z^6)^{\frac{1}{3}}} = \int\frac{z^3dz}{(1-z^6)^{\frac{2}{3}}}$$

or having put z instead of zz this one

$$\int\frac{dz}{(1-z^3)^{\frac{1}{3}}} = \int\frac{zdz}{(1-z^3)^{\frac{2}{3}}},$$

if one puts $z = 1$ after the integration.

COROLLARY 3

§77 Let $m = 1$ and $n = 2$; it will be $\cos\frac{m}{2n}\pi = \frac{1}{\sqrt{2}}$ and hence

$$\int\frac{zdz\sqrt{2}}{(1-z^4)^{\frac{1}{2}}} = \int\frac{zzdz}{(1-z^4)^{\frac{3}{4}}} = \frac{\pi}{2\sqrt{2}}$$

because $\int\frac{zdz}{\sqrt{1-z^4}} = \frac{\pi}{4}$. From Corollary 1 on the other hand it will be

$$\int \frac{zdz\sqrt{2}}{(1-z^4)^{\frac{1}{2}}} = \int \frac{zzdz}{(1-z^4)^{\frac{3}{4}}},$$

which is the same equality.

THEOREM 5

§78 *If one puts $z=1$ after the integrations, it will be*

$$\tan\frac{m}{2n}\pi \cdot \int \frac{z^{n-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}} \cdot \int \frac{z^{2n-m-1}dz}{(1-z^{2n})^{1-\gamma}} = \int \frac{z^{n-1}dz}{(1-z^{2n})^{1-\gamma}} \cdot \int \frac{z^{n+2n\gamma-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}}.$$

PROOF

In § 60 we found that it is

$$\int \frac{z^{n-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}} = \tan\frac{m}{2n}\pi \cdot \int\int \frac{z^{n-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}}.$$

Now let $a=n$, $b=2n$ and $c=\frac{n-m}{2n}$ in Lemma 5 and after the substitution it will be

$$\tan\frac{m}{2n}\pi \cdot \int \frac{z^{n-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}} \cdot \int \frac{z^{2n-m-1}dz}{(1-z^{2n})^{1-\gamma}} = \int \frac{z^{n-1}dz}{(1-z^{2n})^{1-\gamma}} \cdot \int \frac{z^{n+2n\gamma-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}}.$$

Q. E. D.

COROLLARY 1

§79 If one puts $\gamma=1$, because of the two integrable terms it will be

$$\frac{n}{2n-m}\tan\frac{m}{2n}\pi \cdot \int \frac{z^{n-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}} = \int \frac{z^{3n-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}} = \frac{n}{2n-m}\int \frac{z^{n-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}};$$

therefore, it will be

$$\tan\frac{m}{2n}\pi \cdot \int \frac{z^{n-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}} = \int \frac{z^{n-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}},$$

which is the equation found in § 60.

COROLLARY 2

§80 Let $\gamma = \frac{m}{2n}$; it will be

$$\tan\frac{m}{2n}\pi \cdot \int \frac{z^{2n-m-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}} = \int \frac{z^{n+m-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}},$$

and if one puts $\gamma = \frac{1}{2}$, the quadrature of the circle will enter and it will be

$$\int \frac{z^{2n-m-1}dz}{\sqrt{1-z^{2n}}} \cdot \int \frac{z^{n-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}} = \frac{\pi}{2n\tan\frac{m}{2n}\pi} \int \frac{z^{2n-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}} = \frac{\pi}{2n(n-m)\tan\frac{m}{2n}\pi}$$

or

$$\frac{\pi\tan\frac{m}{2n}\pi}{2mn} = \int \frac{z^{n-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}} \cdot \int \frac{z^{n+m-1}dz}{\sqrt{1-z^{2n}}}.$$

COROLLARY 3

§81 Therefore, because it is

$$\frac{\pi}{2mn}\tan\frac{m}{2n}\pi = \int \frac{z^{n-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}} \cdot \int \frac{z^{n+m-1}dz}{\sqrt{1-z^{2n}}}$$

and from § 60 it is

$$\int \frac{z^{n-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}} = \tan\frac{m}{2n}\pi \cdot \int \frac{z^{n-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}},$$

it will be

$$\frac{\pi}{2mn} = \int \frac{z^{n-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}} \cdot \int \frac{z^{n+m-1}dz}{\sqrt{1-z^{2n}}}.$$

Therefore, it is possible to express the product of these integral formulas in terms of the quadrature of the circle in the case $z = 1$.

COROLLARY 4

§82 Let $m = 1$ and $n = 1$; from the preceding corollary it will be

$$\frac{\pi}{2} = \int \frac{dz}{\sqrt{1-zz}} \cdot \int \frac{zdz}{\sqrt{1-zz}} = \frac{\pi}{2}(1-\sqrt{1-zz}),$$

in which case, if it is $z = 1$, the equality is immediately seen.

COROLLARY 5

§83 Let $m = 1$ and $n = 2$; it will be $\tan \frac{m}{2n}\pi = 1$, hence it will be from corollary 2

$$\frac{\pi}{4} = \int \frac{zdz}{(1-z^4)^{\frac{3}{4}}} \cdot \int \frac{zzdz}{\sqrt{1-z^4}},$$

but from the third this equation results

$$\frac{\pi}{4} = \int \frac{zdz}{(1-z^4)^{\frac{3}{4}}} \cdot \int \frac{zzdz}{\sqrt{1-z^4}},$$

which two equations are identical.

COROLLARY 6

§84 Let $m = 2$ and $n = 3$; it will be $\tan \frac{m}{2n}\pi = \sqrt{3}$, hence from Corollary 2 this equation results

$$\frac{\pi}{12} = \int \frac{zzdz}{(1-z^6)^{\frac{2}{3}}} \cdot \int \frac{z^4dz}{\sqrt{1-z^6}}.$$

SCHOLIUM

§85 On can deduce so many theorems of this kind from the integral formulas found for the sine, cosine and tangent by means of the Lemmas 5, 6, 7 and 8, that a whole volume would not suffice to contain them all. But one can take arbitrarily many from the source we opened here. Certainly many cases occur, as we saw, in which one gets either to identical equations or equations of such a kind, which are easily reduced to it, and these cases confirm the truth of the remaining theorems, in which the reason for the equality is not immediately seen, even more. So, if one puts $m = 0$ and $n = 1$ in § 80, it will be $\tan \frac{m}{2n}\pi = \frac{m}{2n}\pi$, since the tangent of a vanishing arc becomes equal to the arc itself; therefore, it will hence be

$$\frac{\pi\pi}{4} = \int \frac{dz}{\sqrt{1-zz}} \cdot \int \frac{dz}{\sqrt{1-zz}},$$

whose validity, since it is $\int \frac{dz}{\sqrt{1-zz}} = \frac{\pi}{2}$ in the case $z = 1$, is clear immediately. Furthermore, comparisons of integral formulas of this kind, which can neither be integrated nor reduced to each other, are the more remarkable, the less there seems to be a way to prove them directly. So the first theorem, which I already found some time ago[6], was remarkable for its simplicity; in this theorem I found that the product of these two integral formulas

$$\int \frac{dz}{\sqrt{1-z^4}} \quad \text{and} \quad \int \frac{z^2 dz}{\sqrt{1-z^4}},$$

the first of which expresses the arc, the other the ordinate on a *curva elastica*, in the case $z = 1$ is equal to the area of the circle whose diameter is $= 1$.

[6]Euler refers to E122 again.